EIGENFUNCTIONS OF THE TWO DIMENSIONAL MOSHINSKY-SZCZEPANIAK OSCILLATOR

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Abstract

While the usual harmonic oscillator potential gives discrete energies in the non-relativistic case, it does not however give genuine bound states in the relativistic case if the potential is treated in the usual way. In the present article, we have obtained the eigenfunctions of the Dirac oscillator in two spatial dimensions, adapting the prescription of Moshinsky.

KeyWords: Two dimensional Dirac Oscillator, eigen functions, Kummer's equation, confluent hypergeometric functions, Moshinsky-Szczepaniak Oscillator.

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The study of quantum harmonic oscillators has received considerable attention and hence it is of intrinsic interest to extend the model to the relativistic domain. Moshinsky[1] formulated a novel prescription of introducing the interaction in the Dirac equation, which, besides the momentum is also linear in co-ordinates and subsequently showed that the non-relativistic form of the interaction reduces to one of the harmonic oscillator type. While in literatuer such a system is called the 'Dirac Oscillator', we suggest the name "Moshinsky-Szczepaniak Oscillator", after its proponents.

Several authors have addressed the Dirac oscillator in one space dimension. Titchmarsh[2] analysed the relativistic harmonic oscillator problem using the Green's function technique. Nogami et.al.[3] have pointed out the interesting differences in the coherent states of the Dirac Oscillator and the non-relativistic harmonic oscillator. The relativistic extension of the one-dimensional oscillator in the Lagrangian formalism omegaas developed by Moreau[4] and others. While Moshinsky[5] has analysed the one-body relativistic oscillator using Group Theory, Villalba[6] has dealt with the angular momentum operator in the Dirac equation. Dominguez-Adame[7] has anaylsed the one-dimensional Dirac Oscillator with a scalar interaction and shown the absence of Klein paradox.

An interesting framework for discussing the Dirac Oscillator is a 2+1 space-time. Presently we explore the two-dimensional Dirac Oscillator and expose some special features not displayed by one-dimensional systems.

The Dirac equation in two dimensions[8] for a free particle would read

$$E\psi = (c\,(\alpha_x p_x + \alpha_y p_y) + \beta m_0 c^2)\psi \ , \tag{1}$$

with m_0 denoting the rest mass of the particle and α_x, α_y and β representing the standard Dirac matrices.

The prescription of Moshinsky is extended to two dimensions and the interaction is introduced as follows

$$p_x \rightarrow p_x - i\beta m_0 w x \tag{2}$$
$$p_y \rightarrow p_y - i\beta m_0 w y \tag{3}$$

The two dimensional Dirac equation may now be written as

$$\left[\beta E + i\hbar c\beta\alpha_x \frac{\partial}{\partial x} + ic\beta\alpha_x\beta m_0 w x + i\hbar c\beta\alpha_y \frac{\partial}{\partial y} + ic\beta\alpha_y\beta m_0 w y - \beta^2 m_0 c^2\right]\psi = 0 \tag{4}$$

It is convenient to introduce the following representation in terms of the Pauli matrices.

$$\alpha_x = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \alpha_y = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \beta = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

Writing $\psi(x) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, the matrix form of Eq. (4) is

$$\begin{pmatrix} E-m_0c^2 & i\hbar c\frac{\partial}{\partial x}+\hbar c\frac{\partial}{\partial y}-icm_0wx-m_0cwy \\ -i\hbar c\frac{\partial}{\partial x}+\hbar c\frac{\partial}{\partial y}-icm_0wx+m_0cwy & -E-m_0c^2 \end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \tag{5}$$

The spinor equation may be written as a system of two first order coupled differential equations

$$\left(E-m_0c^2\right)\psi_1+\left(i\hbar c\frac{\partial}{\partial x}+\hbar c\frac{\partial}{\partial y}-icm_0wx-m_0cwy\right)\psi_2=0 \tag{6}$$

$$\left(-i\hbar c\frac{\partial}{\partial x}+\hbar c\frac{\partial}{\partial y}-icm_0wx+m_0cwy\right)\psi_1-\left(E+m_0c^2\right)\psi_2=0 \tag{7}$$

Equation for ψ_1:

The equation for ψ_1 is obtained by using Eq. (7) in Eq. (6).

On simplification we obtain

$$\left(E^2-m_0^2c^4\right)\psi_1+\Big\{\hbar^2c^2\frac{\partial^2}{\partial x^2}+\hbar^2c^2\frac{\partial^2}{\partial y^2}-m_0^2c^2w^2x^2-m_0^2c^2w^2y^2 +2\hbar c^2m_0w+2m_0c^2wL_z\Big\}\psi_1=0 \tag{8}$$

where $L_z=xp_y-yp_z$ is implied.

Using $p_x=-i\hbar\frac{\partial}{\partial x}$ and $p_y=-i\hbar\frac{\partial}{\partial y}$, it is straightforward to check that the above equation may be written in an elegant form as

$$\left[\left(\frac{p_x^2}{2m_0}+\frac{p_y^2}{2m_0}\right)+\frac{1}{2}m_0w^2\left(x^2+y^2\right)-\hbar w\right]\psi_1=\left(wL_z+\frac{E^2-m_0^2c^4}{2m_0c^2}\right)\psi_1 \tag{9}$$

Apparently, the first two terms are those that appear in the Hamiltonian of a non-relativistic 2D harmonic oscillator. This justifies why the 'potential' is called the Relativistic Oscillator Potential. The fact that Dirac particles remain bound by this interaction suggests the absence of Klein paradox.

Equation (9) may also be written as

$$\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-\frac{m_0^2w^2}{\hbar^2}\left(x^2+y^2\right)+\frac{2m_0}{\hbar^2}\hbar w\right]\psi_1=-\frac{2m_0}{\hbar^2}\left[wL_z+\frac{E^2-m_0^2c^4}{2m_0c^2}\right]\psi_1 \tag{10}$$

It is convenient to use plane polar co-ordinates (ρ,ϕ) to obtain the solution of this equation.

Writing $\rho^2 = x^2 + y^2$, and expressing $\bigtriangledown^2$ as

$$\bigtriangledown^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}$$

the wave function may be expressed as

$$\psi_1 = R(\rho)\,\Phi(\phi) = R(\rho)\,e^{im\phi} \tag{11}$$

where $m = 0, \pm1, \pm2, \pm3, ...$ is the angular momentum quantum number.

Equation (10) takes the form

$$\frac{1}{\rho}\ \frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) - \frac{m^2}{\rho^2}R(\rho) - \frac{m_0^2 w^2\rho^2}{\hbar^2}R(\rho) - \frac{2m_0\ \hbar w}{\hbar^2}R(\rho) +$$

$$\frac{2m_0}{\hbar^2}mw\hbar R(\rho) + \frac{2m_0}{\hbar^2}\left(\frac{E^2 - m_0^2c^4}{2m_0c^2}\right)R(\rho) = 0 \tag{12}$$

Multiplying by ρ^2, the above equation becomes

$$\rho^2\frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} + \left(k^2\rho^2 - m^2\right)R - \frac{m_0^2w^2}{\hbar^2}\rho^4 R = 0. \tag{13}$$

Here,
$$k^2 = \frac{2m_0}{\hbar^2}\left[(m+1)\,\hbar w + \left(\frac{E^2 - m_0^2c^4}{2m_0c^2}\right)\right] \tag{14}$$

It is easy to check that k has the dimensions of inverse length and thus $k\rho$ is a dimensionless parameter.

In what follows we use the notation $z = \frac{m_o w}{\hbar}\rho^2$. In terms of the variable z, the above equation may be written as

$$z^2\frac{d^2R}{dz^2} + z\frac{dR}{dz} + \frac{1}{4}\left\{k_1 z - m^2 - z^2\right\}R(z) = 0 \tag{15}$$

where

$$k_1 = k^2\frac{\hbar}{m_0 w} = 2(m+1) + \frac{(E^2 - m_0^2c^4)}{m_0c^2\hbar w}. \tag{16}$$

While the first term refers to the oscillator part, the second terms refers to the kinetic energy of the particle. We try solutions of the form

$$R(z) = e^{\frac{-z}{2}}z^{\frac{m}{2}}\phi_1(z)\,. \tag{17}$$

The unknown function $\phi_1(z)$ should be a constant for $z \to 0$ and should guarantee normalisation. Equation (15) may be written in the standard form as

$$z\frac{d^2\phi_1}{dz^2} + (b - z)\frac{d\phi_1}{dz} - a\phi_1 = 0 \tag{18}$$

with $b = m + 1$ and $a = \frac{1}{2}\left(m + 1 - \frac{k_1}{2}\right)$. We identify this differential equation as the Kummer's equation[9] whose solution can be expressed in terms of the confluent hypergeometeric functions. The only admissible solution is $M(a, b, z)$. The other linearly independent solution $U(a, b, z)$[10] is rejected since it is irregular at infinity.

We now write the solution as

$$R(\rho) = e^{\frac{-m_0 w}{2\hbar}\rho^2}\left(\frac{m_0 w}{\hbar}\rho^2\right)^{\frac{m}{2}} M\left(\frac{1}{2}(m + 1 - \frac{k_1}{2}), m + 1, \frac{m_0 w}{\hbar}\rho^2\right) \tag{19}$$

The complete wavefunction is written as

$$\psi_1 = Ae^{im\phi}e^{-\frac{m_0 w}{2\hbar}\rho^2}\left(\left(\frac{m_0 w}{\hbar}\right)^{\frac{1}{2}}\rho\right)^m M\left(\frac{1}{2}(m + 1 - \frac{k_1}{2}), m + 1, \frac{m_0 w}{\hbar}\rho^2\right) \tag{20}$$

where A is the normalisation constant. Further, as is well-known, when $a = -n$, the hypergeometric series terminates and defines a finite polynomial of n^{th} degree. Hence

$$\psi_{nm}^{(1)} = Ae^{im\phi}e^{-\frac{m_0 w}{2\hbar}\rho^2}\left(\left(\frac{m_0 w}{\hbar}\right)^{\frac{1}{2}}\rho\right)^m M\left(-n - 1, m + 1, \frac{m_0 w}{\hbar}\rho^2\right) \tag{21}$$

The non-negative integer n, which arises from the boundary condition that the wave function vanishes as $\rho \to \infty$, also quantizes the energy eigenvalues.

Equation for ψ_2

The equation for the small component of the Dirac wave function is obtained in an analogous manner. Eliminating ψ_1 in Eq. (7) first, and then going through similar steps as before, we obtain

$$\left[\left(\frac{p_x^2}{2m_0} + \frac{p_y^2}{2m_0}\right) + \frac{1}{2}m_0 w^2\left(x^2 + y^2\right) + \hbar w\right]\psi_2 = \left[wL_z + \frac{E^2 - m_0^2 c^4}{2m_0 c^2}\right]\psi_2 \tag{22}$$

which resembles Eq. (9). Adapting the same procedure as before it is seen that the eigenfunctions can be expressed as

$$\psi_{nm}^{(2)} = Ae^{im\phi}e^{-\frac{m_0 w}{2\hbar}\rho^2}\left(\left(\frac{m_0 w}{\hbar}\right)^{\frac{1}{2}}\rho\right)^m M\left(-n, m + 1, \frac{m_0 w}{\hbar}\rho^2\right). \tag{23}$$

The Energy Spectrum

The quantisation of energy demands the vanishing of the eigenfunctions at infinity, which comes from the fact that the first argument of $M(a, b, z)$ be equal to a negative integer or zero[8]. We obtain from Eq. (20)

$$\frac{1}{2}\left(m + 1 - \frac{k_1}{2}\right) = -n - 1 \tag{24}$$

or equivalently

$$E^2 - m_0^2 c^4 = 4\,(n + 1)\, m_0 c^2 \hbar w. \tag{25}$$

As is expected for central potentials, the energy eigenvalues are independent of the quantum number m, as a consequence of rotational symmetry.

Non-relativistic limit

The non-relativistic limit is obtained by setting $E = m_0 c^2 + \epsilon_{nr}$ and considering $\epsilon_{nr} << m_0 c^2$. It is straightforward to check that Eq. (25) may be written as

$$E = m_0 c^2 \left[1 + \frac{4\,(n + 1)\,\hbar w}{m_0 c^2}\right]^{\frac{1}{2}}. \tag{26}$$

Taylor expansion of the above would give us

$$E \approx m_0 c^2 + 2\,(n + 1)\,\hbar w - \frac{2\,(n + 1)^2\,\hbar^2 w^2}{m_0 c^2}. \tag{27}$$

It is thus seen that the first term corresponds to the rest energy of the particle, the second term refers to the non relativistic harmonic oscillator energy and the third is the relativistic correction term.

Results and Discussion

The harmonic oscillator problem in quantum mechanics has far-reaching consequences. While the non-relativistic harmonic oscillator problem is addressed by solving the Schrodinger equation directly or by the well-known matrix formulation method or the operator method, the relativistic harmonic oscillator requires an altogether different formalism. The prescription of Moshinsky which describes the one-dimensional Dirac oscillator may well be extended to two space dimension. It is seen that the eigenfunctions are expressed in terms of regular confluent hypergeometric functions, special cases of which are the well known Hermite polynomials. More importantly the eigenenergies have the appropriate non-relativistic limit. The Dirac oscillator is thus a relativistic generalisation of the quantum harmonic oscillator.

Unlike the non-relativistic oscillator, where the energy levels are discrete and equispaced, the relativistic oscillator no doubt has discrete energies, but unevenly spaced levels. Dirac oscillator has potential applications in models of quark confinement in particle physics.

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